WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 4th Semester Examination, 2021

MTMHGEC04T/MTMGCOR04T-MATHEMATICS (GE4/DSC4)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Show that the set of cube roots of unity forms a group with respect to multiplication.

(b) In a group $(G, \circ)$ prove that for all $a, b \in G$, $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$.

(c) When a relation ρ defined on a nonempty set S is said to be an equivalence relation?

(d) Prove that in a commutative group G, the set $H = \{x \in G : x = x^{-1}\}$ forms a subgroup of G.

(e) Show that the group $(Z_4, +)$ is a cyclic group. Find it's generators.

(f) Let R be a ring with 1. Show that the subset $T = \{n1 : n \in \mathbb{Z}\}$ is a subring of R.

(g) Show that the ring $(Z_5, +, -)$ is an integral domain.

(h) Determine whether the permutation f on the group S_5 is odd or even where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}$$

(i) Define index of a subgroup H of a group G. If $G = S_3$ and $H = A_3$, then find the value of $[G : H]$.

2. (a) Let a relation R defined on the set $\mathbb{Z}$ by " aRb if and only if $a-b$ is divisible by 5" for all $a, b \in \mathbb{Z}$. Show that R is an equivalence relation. 4

(b) Which of the following mathematical systems is / are group(s)? 2+2

(i) $(\mathbb{N}, *)$, where $a * b = a$ for all $a, b \in \mathbb{N}$.

(ii) $(\mathbb{Z}, *)$, where $a * b = a - b$ for all $a, b \in \mathbb{Z}$.

3. (a) Let the permutations f and g are the elements of S_5 where 2+2+1

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}.$$ Find fg, gf, f^{-1}.

(b) Let $f: Z \to Z$ is defined by $f(n) = n^2$, $n \in Z$ and $g: Z \to Z$ is defined by $g(n) = 2n$, $n \in Z$. Find the composition of the functions $f \circ g$ and $g \circ f$. 2+1

4. (a) Verify the statement is true or false: In ring R if $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$, then R is a commutative ring. 3

(b) (i) Show that the set $S = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right\}$, $x \neq 0$ is a subgroup of the group of all 2×2 order non-singular real matrices. 2

(ii) Let $(G, \circ)$ be a commutative group and $H = \{a^2 : a \in G\}$, prove that H is sub-group of G. 3

5. (a) Prove that every subgroup of a cyclic group is cyclic. 4

(b) Let G be a group of prime order. Then prove that G is cyclic. 4

6. (a) Find all right cosets of the subgroup $6\mathbb{Z}$ in the group $(\mathbb{Z}, +)$. 4

(b) Let G be a group such that every cyclic subgroup of G is a normal subgroup of G. Prove that every subgroup of G is a normal subgroup of G. 4

7. (a) Let H be the set of all real matrices $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}$. Prove that H is a subset of $GL(2, \boldsymbol{R})$. 4

(b) Find all cyclic subgroups of the group $(S, \cdot)$, where $S = \{1, i, -1, -i\}$. 4

8. (a) Examine if the ring of matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \boldsymbol{R} \right\}$ is a field. 4

(b) Prove that a finite integral domain is a field. 4

9. (a) Show that the ring of matrices $\left\{ \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbf{Z} \right\}$ contains divisors of zeros and does not contain the unity. 4

(b) Prove that the ring $(Z_n, +, \cdot)$ is an integral domain if and only if n is prime. 4

10.(a) Show that $T = \{[0], [5]\}$ is a subring of the ring $\mathbb{Z}_{10}$. 4

(b) Let I and J be ideals of a ring R. Prove that $I + J$ is an ideal of R. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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